Hi AP AB Calculus Class of 2017 – 2018:

In order to complete the syllabus that the College Board requires and to have sufficient time to review and practice for the exam, I am asking you to do a (mandatory) summer project. There are two pieces to the project:

Part I: Analytic Geometry (Conic Sections) Review

Part II: Trigonometric and Logarithmic Review

You can work on it together or by yourself. You can use notes and texts but do not use a calculator. You need to learn the process, not just the answer.

The packet is due on the first class day in September. A late submission receives no credit.

The packet will count as a quiz grade for the first quarter.

Good luck and have a great summer.

Mr. Hyland

Part I OverView of Analytic Geometry

There are two basic problems in Analytic Geometry. First, given a graph, find the equation; second, given an equation, find the graph.

The general form of a quadratic equation, which correspond to conic sections, is as follows: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where the coefficients A, B, C, D, E, and F are real numbers. For the beginning discussion, we will set $B=0$ and investigate the resulting equations.

Case 1) If $A = C$, then there is a possibility of a circle, a point, or no graph.
Case 2) If A does not equal C, but they have the same sign, then there is a possibility of an ellipse, a point, or no graph.
Case 3) If A and C have opposite signs, then there is a possibility of a hyperbola or two lines.
Case 4) If either $A = 0$ or $C = 0$, but not both equal to zero, then the graph will be a parabola.
Case 5) The case where B does not equal zero will be addressed at the end of this section.

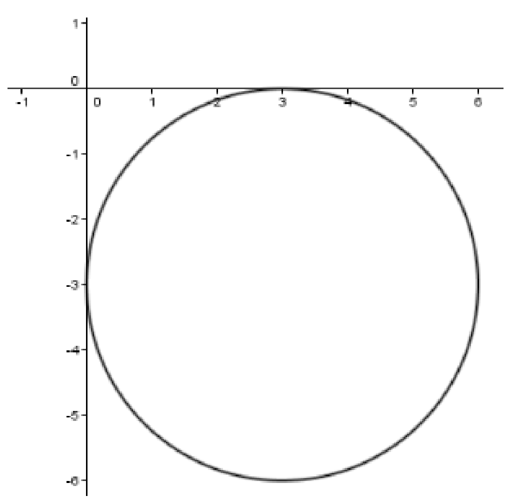
Case 1)

Put $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into a standard form: $(x - h)^2 + (y - k)^2 = r^2$
Definition: A circle is a locus of points (a graph) that are a fixed distance from a fixed point. The fixed distance is the radius (r), and the fixed point is the center located at the point (h, k).

If $r^2 > 0$, we have a circle. If $r^2 = 0$, then we just have the point (h, k). If $r^2 < 0$, then there is no graph.

Example:
Given the general form: $2x^2 + 2y^2 - 12x + 12y + 18 = 0$
The key to putting this equation into standard form is by completing the square for x and for y. For example, at the second step below we have $y^2 + 6y$. To complete the square, divide the coefficient of the linear term (6 in this case) by 2 and square it, then add that number to both sides of the equation.

$x^2 + y^2 - 6x + 6y + 9 = 0$, after dividing through by A (which is 2 in this case)
$x^2 - 6x \quad + y^2 + 6y \quad = -9$, bring the constant to the other side of the equation
$(x^2 - 6x + 9) + (y^2 + 6y + 9) = -9 + 9 + 9$, for both the x and y terms, we squared half of the linear coefficients and added them to both sides of the equation. Now the x and y terms are in the form of perfect squares, and we have the standard form we are looking for:
$(x - 3)^2 + (y + 3)^2 = 9$, here $h = 3$, $k = -3$, $r^2 = 9$
Hence, we have a circle with center at (h, k) = (3, -3), and a radius of 3 (r = square root of 9).



Case 2)

Put $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into a standard form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Definition: An ellipse is the locus of points P(x, y) such that the sum of the distances from P to two fixed points is a constant. The two fixed points are called foci, and the constant is equal to 2a.

In the standard form, a^2 is the larger of the two denominators. If a^2 is under the x variable, then the major axis is horizontal; if a^2 is under the y variable the major axis is vertical. The center is (h, k). The length of the major axis is 2a; the length of the minor axis is 2b. The distance from the center (h, k) to each focus is c, where c is related to a and b as follows: $a^2 - b^2 = c^2$. The foci are on the major axis.

Given: $9x^2 + 25y^2 + 54x - 200y + 256 = 0$

$9x^2 + 54x \quad + 25y^2 - 200y \quad = -256$, rearrange, preparing to complete the squares

$9(x^2 + 6x \quad) + 25(y^2 - 8y \quad) = -256$, factor

$9(x^2 + 6x + 9) + 25(y^2 - 8y + 16) = -256 + 9(9) + 25(16)$, complete the squares, be careful to distribute

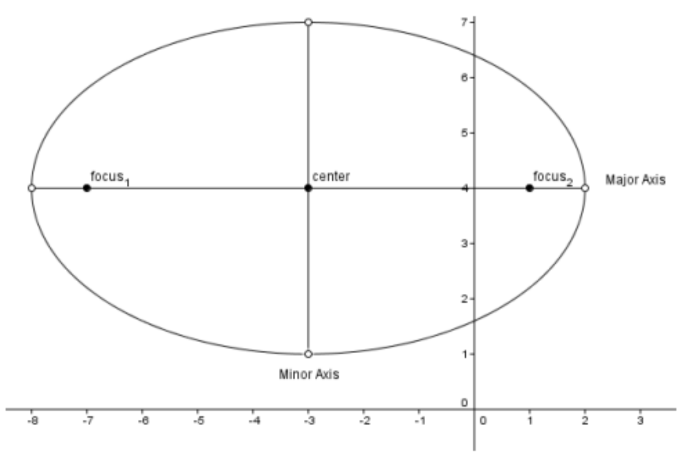
$9(x + 3)^2 + 25(y - 4)^2 = 225$, factor

$\frac{(x+3)^2}{25} + \frac{(y-4)^2}{9} = 1$, divide through by 225, and now we have the equation in standard form

So a = 5, b = 3, the major axis is horizontal since the larger denominator is under the x term. We can calculate c:

$c^2 = a^2 - b^2 = 25 - 9 = 16$, so c = 4.

We have an ellipse with center at (h, k) = (–3, 4). Major axis of length 2a = 10 , minor axis of length 2b = 6 , and foci located c = 4 units left and right of the center, so at (–7, 4) and (1, 4).



Case 3)

2 2

Put $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into a standard form: $\frac{(x - h)}{a_2} - \frac{(y - k)}{b^2} = 1$,
or $\frac{(y - k)_2}{a_2} - \frac{(x - h)^2}{b^2} = 1$

Definition: A hyperbola is the locus of points P(x, y) such that the difference of the distances from P to two fixed points (foci) is a constant (equal to 2a).

The center is (h, k). The distance of each vertex to the center is equal to a. Each focus is a distance c from the center; c is related to a and b as follows: $a^2 + b^2 = c^2$.
If the first term (positive term) is the x term, then the hyperbola opens left and right; if the first term is the y term, the hyperbola opens up and down.

In addition, there are two diagonal asymptotes that the curve approaches as x and y approach infinity. The easiest way to sketch these asymptotes is to sketch a rectangle and draw the diagonals through the rectangle. Take the square root of the value under the y term (regardless of which term is the positive term) and that is the distance of the top and bottom of the rectangle from the center; and the square root of the value under the x term is the distance of the left and right sides of the rectangle from the center. The slopes of the asymptotes can easily be calculated by taking the rise over the run of the two diagonal lines.

Given: $16y^2 – 9x^2 + 54x + 96y – 81 = 0$

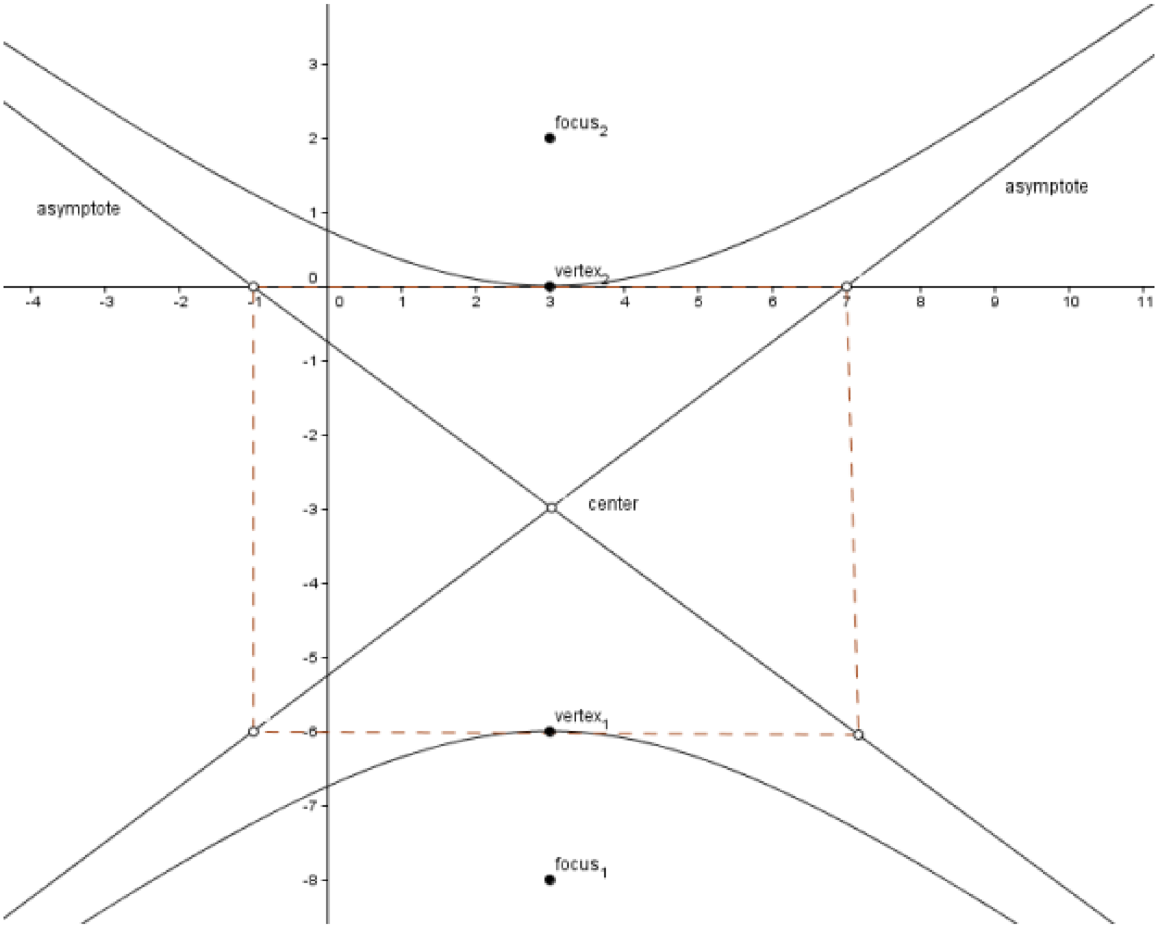
$16y^2 + 96y \quad – 9x^2 + 54x \quad = 81$, rearrange, preparing to complete the squares
$16(y^2 + 6y \quad) \; – \; 9 (x^2 – 6x \quad) = 81$, factor
$16(y^2+ 6y + 9) – \; 9 (x^2 – 6x + 9) = 81 + 16(9) – 9(9)$, complete the squares (be careful to distribute
$16(y + 3)^2 – 9(x – 3)^2 = 144$, factor

$\frac{(y+3)^2}{9} - \frac{(x - 3)^2}{16} = 1$, divide through by 144, and now we have the equation in standard form

So a = 3, b = 4. We can calculate c:
$c^2 = a^2 + b^2 = 9 + 16 = 25$, so c = 5.

We have a hyperbola oriented up/down because the first (positive term) is the y term. The center is at (h, k) = (3, –3). Each vertex is a = 3 units up and down from the center, so at (3, 0) and (3, –6). Each focus is c = 5 units up and down from the center, so at (3, 2) and (3, –8).

Now for the asymptotes, let’s construct the rectangle. The square root of the value under the y-term is 3; the square root of the value under the x-term is 4. So the top of the rectangle is on the line y = 0 (the x-axis), and the bottom is on the line y = –6 (take the y value of the center and add and subtract 3). The left side is on the line x = – 1, and the right side is on the line x = 7 (take the x value of the center and add and subtract 4.



Case 4)

Put $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into a standard form:

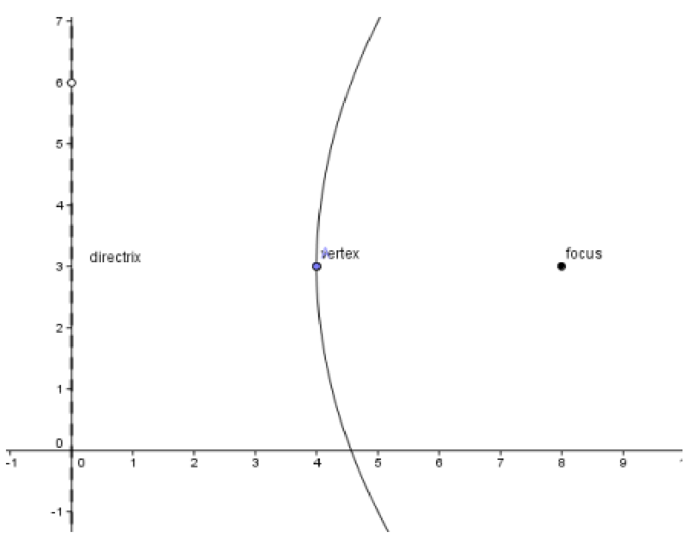
$(4p)(y – k) = (x – h)^2$ or $(4p)(x – h) = (y – k)^2$ (Note: 4p is with the linear term)
Definition: A parabola is the locus of points P (x, y) such that the distance from P to a fixed point (focus) is equal to the distance from that point P to a fixed line, called the directrix, a line outside of the parabola as explained below.

The vertex is (h, k). The focus is inside the parabola a distance p units from the vertex. If the x term is the quadratic (squared) term, the parabola opens up (if $p > 0$) or down (if $p < 0$); in this case the equation is a function (passes the vertical line test), and the directrix is a horizontal line a distance p units from the vertex. If the y term is the quadratic term, the parabola opens right (if $p > 0$) or left (if $p < 0$); in this case the equation is not a function (fails the vertical line test), but we can still graph it and analyze it. In this case, the directrix is a vertical line a distance p units from the vertex.

Given: $y^2 – 16x – 6y + 73 = 0$

$y^2 – 6y \quad = 16x – 73$, rearrange so the quadratic term and its linear term are on the left, everything else is on the right
$y^2 – 6y + 9 = 16x – 73 + 9$, complete the square for the quadratic and add to right side
$y^2 – 6y + 9 = 16x – 64$
$(y – 3)^2 = 16(x – 4)$, factor

Hence we have a parabola opening to the right (since the y term is the quadratic and $4p = 16$, which implies p is positive, and $p = 4$). The vertex is at $(h, k) = (4, 3)$, the focus is inside the parabola, $p = 4$ units to the right of the vertex, at (8, 3). The directrix is outside the parabola $p = 4$ units to the left of the vertex which is the line $x = 0$ (the y-axis).



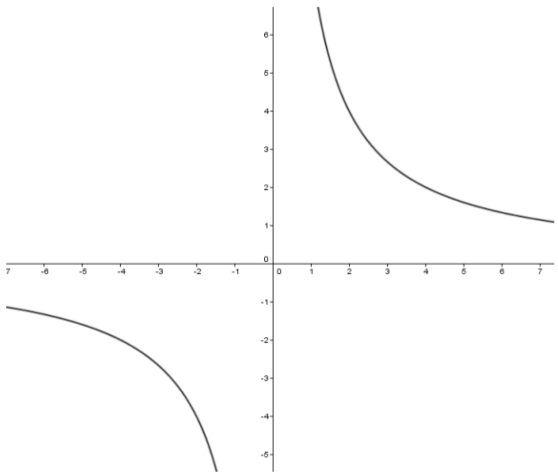
Case 5)

When B does not equal zero, we get a rotation of the graph. Everything else in the figure is relative to the rotation. Let’s look at two simple rotations:

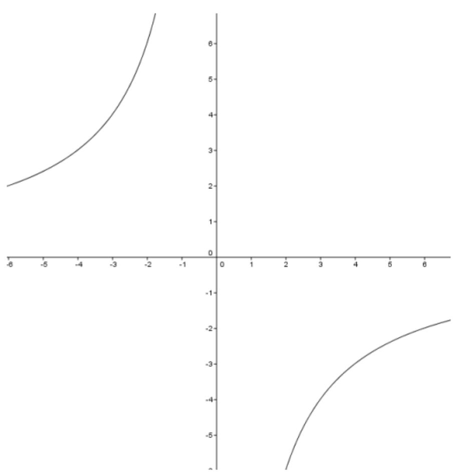
In general form: $xy - 8 = 0$ and $xy + 8 = 0$
In standard form: $xy = 8$ and $xy = -8$

See what happens to the graphs. Instead of being oriented up/down or left/right, they are oriented at an angle relative to the x and y axes:

$xy = 8$



$xy = -8$



Note: Case 5 will not be tested in September. It is included for completeness with Analytic Geometry.

Now try the practice problems due the first day of class in September …

Practice Problems Part I (without a calculator, show ALL work, submit solutions on loose leaf paper):

Problems 1 – 4:
a) Put the General Equation into Standard Form
b) Identify the Conic Section (circle, ellipse, hyperbola, parabola
c) Graph and label the applicable characteristics (vertices, foci, asymptotes, major and minor axes, directrix)

1. $4y = x^2 + 2x - 7$

2. $x^2 + y^2 - 2x + 6y + 5 = 0$

3. $9x^2 + y^2 - 36x - 8y + 43 = 0$

4. $9x^2 - y^2 - 90x - 16y + 125 = 0$

Find the Equation of each of the following conic sections

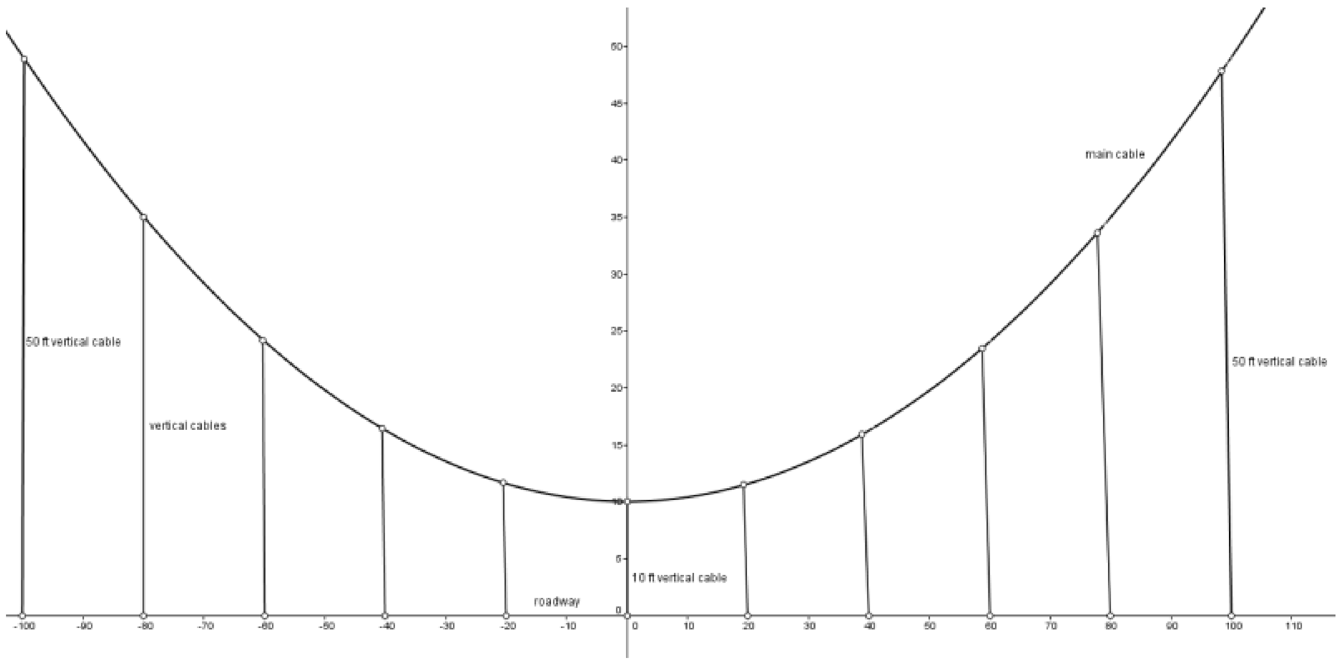
5. Ellipse: vertices (end points of major axis): (–3, –4), (–3, 2); length of minor axis = 4

6. Parabola: focus (0, –5), directrix: the line y = 5

7. Hyperbola: vertices: (1, 0), (–1, 0); foci: (2, 0), (–2, 0)

8. Circle: center (2, 3); point on the circle (10, 9) (hint: use the distance formula to calculate r)

9. The main cable of a suspension bridge is in the form of a parabola and is stabilized by vertical cables extending from the main cable directly down to the side of the roadway. The two vertical cables at the end of the bridge are 50 feet long, and the one in the center of the bridge is 10 feet long. The bridge is 200 feet long. Vertical cables are spaced every 20 feet along the bridge. Calculate the length of each of the vertical cables. (Hint: set the origin at the center of the bridge on the roadway, so the roadway is the x-axis and the origin is on the roadway at the center of the bridge. Write the standard form of the parabola (you know the equation except for 4p), then use your equation and one of the given end lengths (50 feet) to solve for 4p. Now you know the equation of the parabola, so calculate the height of the parabola above the roadway for each of the equally spaced values of x (20 feet apart)). See diagram on the next page.

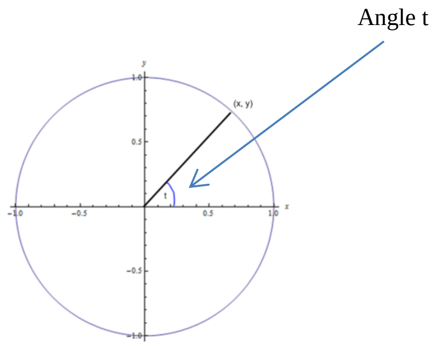


main cable
50 ft vertical cable
vertical cables
50 ft vertical cable
10 ft vertical cable
roadway

End of Part I

Part II

Trigonometry Review



Radian Measure

Let t = the angle in radians as we travel around the unit circle. The cos(t) is the x-coordinate of the point reached by traveling around t radians, starting at the point (1,0); and sin(t) is the y-coordinate. If we travel around the entire unit circle, arriving back at (1,0), we have travelled 2π radians, which implies that 2π radians is equivalent to 360^o or π radians = 180^o. Traveling in the counterclockwise direction is represented by a positive angle; traveling in the clockwise direction is represented by a negative angle. The $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$, $\cos(\pi) = -1$, $\sin(\pi) = 0$, $\sin(-7\pi/2) = 1$. Note that $-7\pi/2$ represents 2 and 3/4 rotations from (1,0) in the negative (clockwise) direction, bringing you to the top of the unit circle where x = 0, y = 1.

The basic equation of the circle with center at the point (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$.The equation of the unit circle is $x^2 + y^2 = r^2$. Since x = cost(t) and y = sin(t), then $\cos^2 t + \sin^2 t = 1$. *This equation is the basic trigonometric identity*, true for any angle t. The important sin and cos values can be derived from a 30-60-90 triangle or a 45-45-90 triangle, applied on the unit circle:

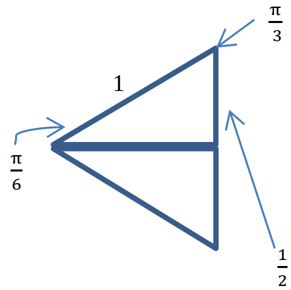
$\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4})$ (by symmetry of a 45-45-90 triangle), so by the basic trigonometric identity:

$\cos^2(\frac{\pi}{4}) + \sin^2(\frac{\pi}{4}) = 2\cos^2(\pi/4) = 1$, so $\cos^2(\pi/4) = \frac{1}{2}$, or $\cos(\pi/4) = \sin(\pi/4) = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

(this is the value in Quadrant I)

Now we have actually found 4 values, one for each quadrant: in Quadrant II the angle $\frac{3\pi}{4}$ has a reference angle with the x-axis of $\frac{\pi}{4}$, and x is negative and y is positive in Quadrant II, so cos is negative and sin is positive. Therefore, $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$, $\sin(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$. Similarly, in Quadrant III, cos is negative and sin is negative, so $\cos(\frac{5\pi}{4}) = \frac{-\sqrt{2}}{2}$, $\sin(\frac{5\pi}{4}) = \frac{-\sqrt{2}}{2}$. And in Quadrant IV, cos is positive and sin is negative, so $\cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$, $\sin(\frac{7\pi}{4}) = \frac{-\sqrt{2}}{2}$. Now what is $\sin(\frac{37\pi}{4})$? Figure out how many 2π full revolutions: $(\frac{37\pi}{4})$ represents 4 full revolutions plus an additional $\frac{5\pi}{4}$, so we are in Quadrant III, at an angle equivalent to $\frac{5\pi}{4}$, so $\sin(\frac{37\pi}{4}) = \frac{-\sqrt{2}}{2}$.

Now for $\frac{\pi}{6}$ and $\frac{\pi}{3}$, use a 30-60-90 triangle.



If you consider that the 30-60-90 triangle is half of an equilateral triangle of side = 1, it easy to see that the side opposite the smaller angle is $\frac{1}{2}$, as shown above. Now use the Pythagorean Theorem and see that the larger leg is $\frac{\sqrt{3}}{2}$.

Now calculate the sin and cos for $\frac{\pi}{6}$ and $\frac{\pi}{3}$, using SOH CAH TOA. What is the sin($\frac{53\pi}{6}$)? Calculate how many 2π full revolutions are in $\frac{53\pi}{6}$: $\frac{53\pi}{6} = \frac{48\pi}{6} + \frac{5\pi}{6}$, so there are 4 full revolutions and an additional $\frac{5\pi}{6}$, which is in Quadrant II, so sin is positive. At an angle of $\frac{5\pi}{6}$, the reference angle (formed with the negative x-axis) is $\frac{\pi}{6}$. Therefore, using the 30-60-90 triangle above, $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so $\sin(\frac{53\pi}{6}) = \frac{1}{2}$.

The trigonometric ratios:

$$\tan(t) = \frac{\sin(t)}{\cos(t)}\ ,\quad \cot(t) = \frac{\cos(t)}{\sin(t)}\ ,\quad \sec(t) = \frac{1}{\cos(t)}\ ,\quad \csc(t) = \frac{1}{\sin(t)}$$

Calculate cot($\frac{529\pi}{3}$) without a calculator (this should take less than a minute).

Trigonometric Identities:

Show $\frac{1+\sin(t)}{\cos(t)} + \frac{\cos(t)}{1+\sin(t)} = 2\sec(t)$

This is a possible multiple choice AP question. They would have the right side as one of 5 choices. Algebraically, manipulate the fractions using a least common denominator to arrive at a simpler equivalent:

$$\frac{(1+\sin(t))(1+\sin(t))+\cos(t)\cos(t)}{\cos(t)(1+\sin(t))} = \frac{(1+2\sin(t)+\sin^2(t)+\cos^2(t))}{\cos(t)(1+\sin(t))} = \frac{1+2\sin(t)+1}{\cos(t)(1+\sin(t))} = \frac{2(1+\sin(t))}{\cos(t)(1+\sin(t))} =$$
$$\frac{2}{\cos(t)} = 2\sec(t)$$

Here are some identities to know:

cos(a+b) = cos(a)cos(b) – sin(a)sin(b)
cos(a–b) = cos(a)cos(b) + sin(a)sin(b)
sin(a+b) = sin(a)cos(b) + cos(a)sin(b)
sin(a–b) = sin(a)cos(b) – cos(a)sin(b)

From these identities, you can easily derive the following (let a=b, and use the basic trigonometric identity):
cos(2a) = $cos^2(a) - sin^2(a) = 2cos^2(a) - 1$
sin(2a) = 2sin(a)cos(a)

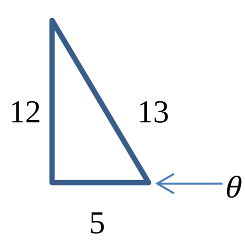
You can use these formulas to calculate trigonometric values for other angles besides 30, 45, and 60 degrees.

What is the cos(75°)? cos(75°) = cos(30° + 45°) = cos(30°) cos(45°) – sin(30°)sin(45°) = $\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$
How would you find cos(285°)? Turn it into this problem: cos(285°) = cos(225° + 60°). Note that 225° has a reference angle of 45° and is located in Quadrant 3.

Inverse Trigonometric Functions
If sin(a) = c, then arcsin(c) = a or $\sin^{-1}(c) = a$. For example, $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$. Think of it as the angle whose sin is $\frac{1}{2}$, which is the angle $\frac{\pi}{6}$.

Now let's think deeper. What is the value of $\sin(\cos^{-1}(\frac{1}{2}))$? First of all, what angle has cos equal to $\frac{1}{2}$? It would be either $\frac{\pi}{3}$ or $\frac{5\pi}{3}$, both of which have cos equal to $\frac{1}{2}$. Unless the quadrant is stated explicitly, the value $\cos^{-1}$ is assumed to be in Quadrant I or II (out of Quadrant I for positive values and out of Quadrant II for negative values). (For $\sin^{-1}$ or $\tan^{-1}$, the values are assumed to be from Quadrant I or IV). So we choose $\frac{\pi}{3}$, and the problem reduces to $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

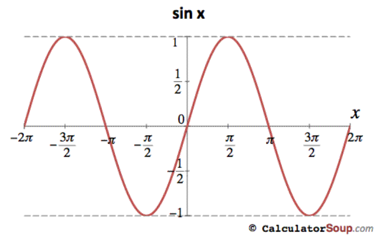
What is $\cos(\sin^{-1}(\frac{12}{13}))$? Draw a picture:

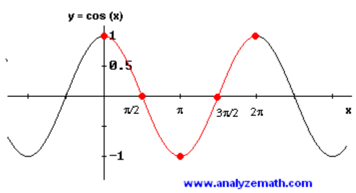


How do you know the sides of the triangle? SOH CAH TOA ! In this case, SOH = $\frac{12}{13}$, so 13 is the length of the hypotenuse and 12 is the length of the leg opposite the angle of interest, θ. By the Pythagorean Theorem, calculate the length of the other leg = 5. Using the diagram, the question is really asking what is cos θ, and using CAH, it is $\frac{5}{13}$.

Graphing Trigonometric Functions
Remember the values along the x-axis are angles, in radians; and the values on the y-axis are real numbers.

y = sin x





Try graphing y = $\frac{1}{4}$sin x. The coefficient of the trig function is called the amplitude; it tells you how high and low the curve goes. In this case, the graph would have the same period, 2π, as the first graph above, but its maximum value is $\frac{1}{4}$ and its minimum value is $\frac{-1}{4}$. Now try graphing y = sin(2x) or y= sin($\frac{1}{4}x$). The maximum and minimum are 1 and -1 because the amplitude is 1, but the period has changed. The coefficient in front of the x determines the period. In the graphs above, a single cycle is completed in 2π radians. But for y = sin(2x), the cycles are completed twice as fast; one cycle is completed in π radians.

For the general form y = Asin(Bx), the amplitude is A, the maximum is A, and the minimum is –A; the period is determined by $\frac{2\pi}{B}$. What is the amplitude and period of the function y = 3sin(7x)? The amplitude is 3 and the period is $\frac{2\pi}{B}$, which is $\frac{2\pi}{7}$ in this case.

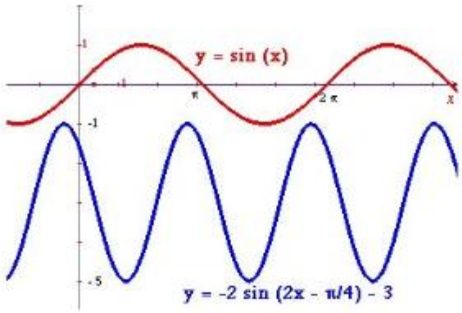
Now we can translate trigonometric functions in the same way any function is translated. The function y = 3sin(7x) + 2 has the same period and amplitude as above, but it has been translated up 2 units. Therefore, the maximum is 3 + 2 = 5, and the minimum is –3 + 2 = –1. Now let's translate the graph left or right. The function y = sin(x – 2) still has a period of 2π, and an amplitude of 1, with a maximum value of 1 and a minimum value of –1; however, we are subtracting two units from x, which translates the graph 2 units to the right.

The general form of the equation is: y = Asin(B(x – h)) + k , where A is the amplitude, the period is $\frac{2\pi}{B}$, the graph is translated h units to the right or left, and k units up or down. Note that B should always be factored out, so the coefficient of x is 1.

For example, how would we graph:

$$y + 3 = –2\sin(2x – \frac{\pi}{4})$$

It is simple if you work step by step. Let's rewrite it as: $y = –2\sin(2(x – \frac{\pi}{8})) – 3$, where we have isolated y and factored out B, so x has a coefficient of 1. Note the B is also factored out of the h because of the distributive property. So we have an amplitude of 2 (the negative sign flips the sin function so it starts by going down instead of up from the origin), a period of $\frac{2\pi}{2} = \pi$, a translation to the right by $\frac{\pi}{8}$ units and a translation down by 3 units. The graph looks like this (compared with the graph of y = sinx):



Note the general effects on f(x) (not just for trig functions)

Function Translations:

f(x - h) + k, shifts the function right or left h units and up or down k units

– f(x) flips the graph over the x-axis

f(-x) flips the graph over the y-axis.

Even and Odd Functions:

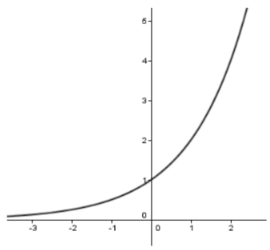
Even: f(x) = f(–x)

Odd: f(–x) = –f(x)

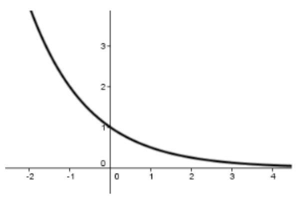
Logarithms

Try graphing $y = 2^x$, $y = 3^x$, or in general $y = a^x$, where a is a positive constant. These graphs are asymptotic to the negative x-axis, slowly increase through the point (0, 1), then quickly increase to $+\infty$ as x approaches $+\infty$. If a is strictly between 0 and 1, the curve is asymptotic to the positive x-axis and increases to $+\infty$ as x approaches $-\infty$.

$f(x) = 2^x$



$f(x) = (\frac{1}{2})^x$



Note that $y = (\frac{1}{2})^x$ is the same as $y = 2^{-x}$ because of the effect of negative exponents ($2^{-1} = \frac{1}{2}$). So, thinking in terms of transformations of functions, if $f(x) = 2^x$, then in the second graph we really have $f(-x)$, which flips the graph over the y-axis.

Given $y = 2^x$, let's find its inverse. *Remember how to find an inverse: exchange the roles of x and y, and solve for y.* So we have $x = 2^y$, but how do we isolate y in this case? We take the logarithm of both sides. Logarithms are inverses of exponentials. So we have $\log_2 x = y$, or $y = \log_2 x$ (read as y = log base 2 of x). In general, the exponential form $c = b^a$ implies the log form $a = \log_b c$. Therefore, the expression $3 = \log_{10} 1000$ can be thought of as "10 (the base) to the power 3 equals 1000 because *logs are exponents*.

The following properties will save you a lot of work – they simply come from the laws of exponents, because *logs are exponents* (notice above where $c = b^a$ implies $a = \log_b c$, when we took the log of both sides, the log base b of b^a is just a, the exponent!).

$\log_b(ac) = \log_b a + \log_b c$ (log of a product is the sum of the logs)

$\log_b(\frac{a}{c}) = \log_b a - \log_b c$ (log of a quotient is the difference of the logs)

$\log_b(a^c) = (c)\log_b a$ (power rule)

and therefore, $\log_b(\sqrt[c]{a}) = \log_b(a^{1/c}) = \frac{1}{c}\log_b(a)$

and $\log_b(b^x) = x \log_b b = x$, since $\log_b b = 1$

If $\log_b a = \log_b c$, then $a = c$

$b^{\log_b(x)} = x$ This property is from the fact that exponentials and logs are inverses of each other.
What happens when we compose a function f(x) with its inverse $f^{-1}(x)$?
$f(f^{-1}(x)) = x$.
$\log_b 1 = 0$

Change of base formula from base a to base b:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

As a mnemonic, notice that the <u>subscript</u> a ends up <u>down</u> in the <u>denominator</u>, and the <u>argument</u> x is <u>up</u> in the <u>numerator</u>.

This formula allows you to change from any base to base 10, since that is the base in your calculator. There is another base with an enormous number of applications: base e.

If the base is not specified, the base is assumed to be base 10. log 100 = 2.

Base e
The number e is an irrational number, like π.
e = 2.71828182845904523536028747135266…
Like π, e exists in nature; If you hang a string or a chain between two points with some slack and let it hang under its own weight, the path assumed by the string can be described by a complicated equation that depends on e. Euler found this constant (hence the letter "e"). There are several methods to compute e; the simplest is: $e = \lim_{x\to 0}(1+x)^{1/x}$. For example, when x = .0001, $(1+x)^{1/x} = 2.7182$.

Consider the hyperbola $y = \frac{1}{x}$ and look at the part in the first quadrant. It is a smooth curve, asymptotic to both the x-axis and y-axis, and goes through the point (1, 1). Now draw a vertical line at x = 1 and another vertical line at x = e (approximately 2.7). Shade in the area under the curve but above the x-axis between these two line segments. This area equals exactly 1 square unit.

Now plot $y = 3^x$, note that the curve is continuous and always increasing. If you look at the slope of the curve at various points, it is always positive and increases as you go to the right. In the negative direction, the slope approaches zero (horizontal slope) as the curve approaches the x-axis, where y = 0. It appears that the slope of an exponential curve is an exponential itself. However, there is one particular curve that goes a couple of steps further. If $y = e^x$, then the slope as a function of x is also equal to e^x. The slope at any value of x is exactly the same as the value of the original function at that value of x. So the slope at the point (5, e^5) is e^5. This curve is the only curve (except y = 0) whose slope is exactly equal to the value of the function.

The logarithm base "e" is called the natural log (ln). It is written y = ln x (read y = "l" "n" of x). Fortunately, all the properties of logs apply to this particular log.
The functions $y = e^x$ and y = ln x are very important in calculus!

Examples:

Convert from exponential form to log form: $25 = 5^2$ converts to: $\log_5(25) = 2$
Convert from log form to exponential form: $\log_{1/2}(8) = -3$ converts to $8 = (\frac{1}{2})^{-3}$

Sometimes this process requires some manipulation first:

$4^{\log_2(3)}$, is not yet in the right form. Since the base of the log is 2, we would like this problem to be written as 2 to some exponent. Well, 4 is 2^2 , so we can rewrite it as $2^{2(\log_2(3))}$, now by the power rule this is equal to $2^{(\log_2(9))} = 9$.

The following practice problems are due the first day of class in September.

Practice Problems Part II (without a calculator, show ALL work, submit solutions on loose leaf paper):

Evaluate (all numbers must be exact):

1. $\cos(\frac{35\pi}{4})$
2. $\sin(\frac{-397\pi}{6})$
3. $\csc(\frac{5\pi}{4})$
4. $\tan(\frac{-5\pi}{3})$
5. $\sin(330^o)$
6. $\sin(-135^o)$
7. $\cos(105^o)$
8. $\cos(435^o)$
9. $\sin(-\frac{\pi}{12})$
10. $\tan(195^o)$
11. $\sin(\pi - x)$
12. $\cos(\frac{\pi}{2} - x)$
13. $\sin(\frac{\pi}{2} - x)$
14. $\cos(-x)$ (hint: $-x = 0 - x$)
15. $\sin(-x)$

Prove (select one side of the equation only to manipulate and manipulate it until it exactly equals the other side:

16. $\tan(x) + \cot(x) = \sec(x)\csc(x)$
17. $\tan(x)\sin(x) = \sec(x) - \cos(x)$
18. $cot^2(x) + 1 = csc^2(x)$
19. $sin^4(x) - cos^4(x) = sin^2(x) - cos^2(x)$
20. $(1 + \sin(x))^2 = 2(1 + \sin(x)) - cos^2(x)$
21. $\frac{\sin(x)+\cos(x)}{\sec(x)+\csc(x)} = \frac{\cos(x)}{\csc(x)}$
22. $\frac{1}{1+\sin(x)} + \frac{1}{1-\sin(x)} = 2sec^2(x)$
23. $\sqrt{\frac{1-\sin(x)}{1+\sin(x)}} = |\sec(x) - \tan(x)|$

Solve (provide exact answers):

24. $\arcsin(0)$
25. $\sin^{-1}(\frac{1}{2})$
26. $\arcsin(-\frac{1}{2})$
27. $\arctan(1)$
28. $\cos^{-1}(\frac{\sqrt{3}}{2})$
29. $\sec(\arcsin(\frac{2}{3})$
30. $\tan(\cot^{-1}(5))$
31. $\sin(2\cos^{-1}(\frac{1}{5}))$
32. $\tan(\cos^{-1}(\frac{-1}{4}))$

Graph (without a calculator):

33. $y + 3 = 5 + 3\sin(\frac{2}{5}x + \frac{6\pi}{5})$
34. $y = \frac{3}{10}\cos(3(x + \frac{\pi}{4})) + 3$

Simplify:

35. $(x^3)(2x)(xy^2)^{-1}$
36. $(3^5)(3^{-2})$
37. $(2^3)(2^7)$
38. $(9^{1/2}) - (9^{-1/2})$

Solve:

39. $2^x = 4^{(2x-1)}$
40. $x^{(-2/3)} = 4$
41. $(x + 3)^{1/2} = (\frac{1}{8})^{1/3}$

Rewrite in opposite form (exponential form to log form, or log form to exponential form):

42. $\log_3(9) = 2$
43. $\log_7(7) = 1$
44. $\log_{1/2}(x) = -3$
45. $125 = 5^3$
46. $\frac{2}{3} = (3/2)^{-1}$
47. $\frac{1}{6} = (1/36)^{1/2}$
48. $\log 1000 = 3$
49. $\ln(e^4) = 4$

Continued on next page…

Solve:

50. $\log_x(8) = 3$

51. $\log_x(\frac{1}{4}) = -\frac{1}{2}$

52. $\log_3(x) = -2$

53. $\log_4(x) = -\frac{1}{2}$

54. $\log_3(\frac{1}{9}) = x$

55. $\log_2(8) - \log_{1/2}(8) = x$

56. $3\log(4) - 2\log(8) = x$

Evaluate:

57. $4^{2\log_4(3)}$

58. $e^{\ln(e)}$

59. $e^{3\ln(2)}$

60. $e^{5\ln(e)}$

61. $8^{\log_2(5)}$

62. $16^{\log_4(2)}$

63. $e^{\log_{e^2}(3)}$

64. $e^{3\ln(2)}$